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LETTER TO THE EDITOR

Relativistic Calogero–Sutherland model: spin generalization, quantum affine symmetry and dynamical correlation functions

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Abstract. Spin generalization of the relativistic Calogero–Sutherland model is constructed by using the affine Hecke algebra and shown to possess the quantum affine symmetry $U_q(\widehat{gl}_2)$. The spinless model is exactly diagonalized by means of the Macdonald symmetric polynomials. The dynamical density–density correlation function, as well as the one-particle Green function, are evaluated exactly. We also investigate the finite-size scaling of the model and show that the low-energy behaviour is described by the $C = 1$ Gaussian theory with a new selection rule. The results indicate that the excitations obey the fractional exclusion statistics and also exhibit the Tomonaga–Luttinger liquid behaviour.

Recently, Yangian symmetry has been extensively studied [1–3] in relation to the Calogero–Sutherland model (CSM) [4], the Haldane–Shastry model (HSM) [5] as well as conformal field theory (CFT). In particular, it is remarkable that a new structure of CFT called spinon structure has been understood based on this symmetry [3]. This motivated the authors in [6] to analyse the analogous structure of the level-1 integrable highest weight modules of the quantum affine algebra $U_q(\widehat{sl}_2)$. These modules and their duals are known to provide a space of states of the XXZ spin chain in the antiferromagnetic regime [7]. The level-0 action of $U_q(\widehat{sl}_2)$ in the level-1 modules has been shown to play the same role as the Yangian in CFT. Namely the level-1 modules are completely reducible with respect to the level-0 action. However, no physical models related to this level-0 symmetry have been discussed.

One purpose of this letter is to propose a model having this symmetry. We consider the trigonometric limit of the Ruijsenaars–Schneider model (RSM) [8], which may be considered as a relativistic extension of the CSM, and show that the spin generalization of the model possesses a desired symmetry. This consideration should deepen our understanding of finite-dimensional quantum integrable systems from the quantum group point of view. In addition, recently the RSM itself has been shown to have wide connections with various subjects, such as the sine–Gordon theory [8, 9], the G/G gauged Wess–Zumino–Witten model on a cylinder with a certain Wilson line insertion [10], 2D Toda chains [8, 11] and the integrable structure of the four-dimensional supersymmetric Yang–Mills theory [12]. In the second half of this paper, we investigate physical properties of the trigonometric RSM.

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Let us start with the definition of the trigonometric RSM. Let θ_j $(j = 1, 2, \ldots, N)$ be the rapidity variables and x_j be their canonically conjugate variables. We impose the canonical commutation relations $[x_j, \theta_l] = \mathrm{i}\delta_{j,l}$ with $\hbar = 1$ and use the representation $\theta_l = -\mathrm{i}\partial/\partial x_l$. The model is described by the following Hamiltonian H and momentum operator P

$$H = \frac{c^2}{2}(H_{-1} + H_1) \qquad P = \frac{c}{2}(H_{-1} - H_1) \tag{1}$$

with N independent integrals of motion H_k (or H_{-k}) $(k = 1, 2, \ldots, N)$

$$H_{\pm k} = \sum_{\substack{I\subset\{1,2,\ldots,N\}\\ |I|=k}} \prod_{\substack{i\in I\\ j\notin I}} \left(\frac{\sin\frac{1}{2}\alpha(x_i - x_j \mp \mathrm{i}g/c)}{\sin\frac{1}{2}\alpha(x_i - x_j)}\right)^{1/2} \mathrm{e}^{-1/c\sum_{i\in I}\theta_i}$$
$$\times \prod_{\substack{l\in I\\ m\notin I}} \left(\frac{\sin\frac{1}{2}\alpha(x_m - x_l \mp \mathrm{i}g/c)}{\sin\frac{1}{2}\alpha(x_m - x_l)}\right)^{1/2} \tag{2}$$

where c is the speed of light, $g \in \boldsymbol{Q}$ is the coupling constant and $\alpha \in \boldsymbol{R}_{>0}$. We normalize the mass equal to one. The model possesses the Lorentz boost generator $B = -(1/c)\sum_{i=1}^N x_i$, and is Poincaré invariant in the sense that the operators H, P and B satisfy the Poincaré algebra

$$[H, P] = 0 \qquad [H, B] = \mathrm{i}P \qquad [P, B] = \mathrm{i}H/c^2. \tag{3}$$

In the non-relativistic limit $c \to \infty$, we recover the Hamiltonian of the CSM

$$\lim(H - Nc^2) = -\sum_{j=1}^N \frac{1}{2}\left(\frac{\partial}{\partial x_j}\right)^2 + \frac{g(g-1)}{4}\sum_{1\leqslant j<k\leqslant N} \frac{\alpha^2}{\sin^2\frac{1}{2}\alpha(x_j - x_k)}$$

with identification $\alpha = 2\pi/L$, where L is the length of a ring on which particles are confined.

It is also known that the integrals of motions H_k can be gauge transformed to the Macdonald operators [13]. Let us define new parameters $p = \mathrm{e}^{-\alpha/c}$, $t = p^g$ and new variables $z_j = \mathrm{e}^{\mathrm{i}\alpha x_j}$, $p^{\pm\vartheta_j} = \mathrm{e}^{\mp(\alpha/c)z_j\partial/\partial z_j}$. Notice the relation $p^{\pm\vartheta_j}z_j = p^{\pm 1}z_j$. Then, by using the function

$$\Delta = \prod_{\substack{j,k=1\\ j\neq k}}^N \frac{(z_j/z_k;\, p)_\infty}{(tz_j/z_k;\, p)_\infty} \tag{4}$$

with $(x;\, p)_\infty = \prod_{n=0}^\infty(1 - xp^n)$, one has [14]

$$\Delta^{-1/2}H_{\pm k}\Delta^{1/2} = t^{\mp k(N-1)/2}D_k(p^{\pm 1}, t^{\pm 1}). \tag{5}$$

Here $D_k(p, t)$ are the Macdonald operators defined by [13]

$$D_k(p, t) = t^{k(k-1)/2}\sum_{\substack{I\subset\{1,2,\ldots,N\}\\ |I|=k}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tz_i - z_j}{z_i - z_j}\prod_{i\in I} p^{\vartheta_i}. \tag{6}$$

Now let us discuss a spin generalization of the model and clarify its quantum affine symmetry. The model we will consider is essentially the trigonometric model discussed by Bernard *et al* [2], but it has never been connected to the relativistic CSM. Let us consider the trigonometric solution $\bar{R}(z)$ [7] of the Yang–Baxter equation and the operator $L_{0i}(z)$ $(i = 1, 2, \ldots, N)$ defined by

$$L_{0i}(z) = \frac{1 - q^2 z}{(1 - z)q}\bar{R}_{0i}(z) = \frac{zS_{0i}^{-1} - S_{0i}}{1 - z}P_{0i} \tag{7}$$

where q $(\neq 0)$ is a complex parameter, $P_{ij}(v_i \otimes v_j) = v_j \otimes v_i$ with $v_j = v_+$ or v_- is a basis of two-dimensional vector spaces V_j $(j = 0, 1, \ldots, N)$ and

$$S = \begin{pmatrix} -q^{-1} & & & \\ & q - q^{-1} & -1 & \\ & -1 & 0 & \\ & & & -q^{-1} \end{pmatrix}. \tag{8}$$

We regard $L_{0i}(z)$ as a linear operator on $V_0 \otimes V_i$. Note that the operators $S_{jj+1}(j = 1, 2, \ldots, N-1)$ satisfy the Hecke algebra relations.

$$\begin{aligned} &S_{jj+1} - S_{jj+1}^{-1} = q - q^{-1} \\ &S_{jj+1}S_{kk+1} = S_{kk+1}S_{jj+1} \qquad |j-k| > 1 \\ &S_{jj+1}S_{j+1j+2}S_{jj+1} = S_{j+1j+2}S_{jj+1}S_{j+1j+2}. \end{aligned} \tag{9}$$

Define the monodromy matrix $L_0(z)$ by

$$L_0(z) = L_{01}(z)L_{02}(z)\ldots L_{0N}(z). \tag{10}$$

Then the operators $\bar{R}(z)$ and $L_0(z)$ satisfy the relation

$$\bar{R}_{00'}(z/z')L_0(z)L_{0'}(z') = L_{0'}(z')L_0(z)\bar{R}_{00'}(z/z'). \tag{11}$$

We use this relation to realize the quantum affine symmetry $U_q(\widehat{gl}_2)$ as well as to define an integrable spin generalization of the model. For this purpose, we introduce the notion of affine Hecke algebra $\hat{H}_N(q)$ [2]. The algebra $\hat{H}_N(q)$ is generated by g_{jj+1} $(j = 1, 2, \ldots, N-1)$ and y_j $(j = 1, 2, \ldots, N)$ with relations (9) for g_{jj+1} and

$$\begin{aligned} &y_j y_k = y_k y_j \qquad g_{jj+1}y_j g_{jj+1} = y_{j+1} \\ &[g_{jj+1}, y_k] = 0 \qquad (j, j+1 \neq k). \end{aligned} \tag{12}$$

We use the following representation of $\hat{H}_N(q)$ [6]:

$$\begin{aligned} &g_{jk}^{\pm 1} = \frac{qz_j - q^{-1}z_k}{z_j - z_k}(1 - K_{jk}) - q^{\mp 1} \\ &y_j = r_{jj+1}^{-1}\ldots r_{jN}^{-1}p^{\vartheta_j}r_{1j}\ldots r_{j-1j} \end{aligned}$$

with $K_{jk}f(\ldots, z_j, \ldots, z_k, \ldots) = f(\ldots, z_k, \ldots, z_j, \ldots)$ and $r_{jk} = K_{jk}g_{jk}$.

Since the operators y_j $(j = 1, \ldots, N)$ commute with each other, the 'quantized' monodromy matrix [2]

$$\hat{L}_0(z) = L_{01}(zy_1)\ldots L_{0N}(zy_N) \tag{13}$$

also satisfies relation (11). Consider the formal expansion of $\hat{L}_0(z)$ in $z^{\pm 1}$ and define

$$\hat{L}_0^{\pm}(z) = \sum_{\pm n \geqslant 0} z^n \begin{pmatrix} l_{11}^{\pm}[n] & l_{12}^{\pm}[n] \\ l_{21}^{\pm}[n] & l_{22}^{\pm}[n] \end{pmatrix}. \tag{14}$$

From (7) and (11), we have the relations $l_{21}^{+}[0] = l_{12}^{-}[0] = 0$ and $l_{jj}^{+}[0]l_{jj}^{-}[0] = 1$ $(j = 1, 2)$ as well as

$$\bar{R}_{00'}(z/z')\hat{L}_0^{\pm}(z)\hat{L}_{0'}^{\pm}(z') = \hat{L}_{0'}^{\pm}(z')\hat{L}_0^{\pm}(z)\bar{R}_{00'}(z/z') \tag{15}$$

$$\bar{R}_{00'}(z/z')\hat{L}_0^{+}(z)\hat{L}_{0'}^{-}(z') = \hat{L}_{0'}^{-}(z')\hat{L}_0^{+}(z)\bar{R}_{00'}(z/z'). \tag{16}$$

Now let $\mathcal{F}_N$ be the space of vectors $v \in \{f(z_1, z_2, \ldots, z_N) \otimes V^{\otimes N}\}$ satisfying

$$(g_{jj+1} - S_{jj+1})v = 0 \qquad j = 1, 2, \ldots, N-1. \tag{17}$$

Relations (15) and (16) define a level-0 representation $\pi^{(N)}$ of $U_q(\widehat{gl}_2)$ on $\mathcal{F}_N$ [6]. From (13) and (14), we obtain its explicit form as

$$\pi^{(N)}(e_0) = \sum_{j=1}^{N} y_j^{-1} q^{h_1} \otimes \cdots \otimes q^{h_1} \otimes \overset{j}{\check{f}_1} \otimes q^{h_2} \otimes \cdots \otimes q^{h_2}$$

$$\pi^{(N)}(f_0) = \sum_{j=1}^{N} y_j q^{-h_2} \otimes \cdots \otimes q^{-h_2} \otimes \overset{j}{\check{e}_1} \otimes q^{-h_1} \otimes \cdots \otimes q^{-h_1}$$

$$\pi^{(N)}(e_1) = \sum_{j=1}^{N} q^{h_2} \otimes \cdots \otimes q^{h_2} \otimes \overset{j}{\check{e}_1} \otimes q^{h_1} \otimes \cdots \otimes q^{h_1}$$

$$\pi^{(N)}(f_1) = \sum_{j=1}^{N} q^{-h_1} \otimes \cdots \otimes q^{-h_1} \otimes \overset{j}{\check{f}_1} \otimes q^{-h_2} \otimes \cdots \otimes q^{-h_2}$$

$$\pi^{(N)}(q^{\pm h_j}) = q^{\pm h_j} \otimes \cdots \otimes q^{\pm h_j} \qquad j = 1, 2$$

where on the right-hand side $e_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f_1 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Note that the quantum determinant q-det $\hat{L}_0(z)$ commutes with the level-0 action $U_q(\widehat{gl}_2)$ and is an appropriate object to construct a desired model. Direct calculation shows

$$q\text{-det}\, \hat{L}_0(z) = q^N \prod_{j=1}^{N} \frac{(1 - q^{-1} y_j z^{-1})}{(1 - q y_j z^{-1})}. \tag{18}$$

Expanding q-det $\hat{L}_0(z)$ in the power of z^{-1}, one gets the commuting family of N-independent operators

$$\sum_{i_1 < \cdots < i_k} y_{i_1} \ldots y_{i_k} \qquad (k = 1, 2, \ldots, N). \tag{19}$$

Now we define a model on $\mathcal{F}_N$ by the following Hamiltonian $\hat{h}$ and momentum operator $\hat{p}$:

$$\hat{h} = \frac{c^2}{2} \sum_{j=1}^{N} (y_j^{-1} + y_j) \qquad \hat{p} = \frac{c}{2} \sum_{j=1}^{N} (y_j^{-1} - y_j). \tag{20}$$

Defining also the operator $\hat{b} = -(\mathrm{i}/\alpha) \sum_{j=1}^{N} \ln z_j$, one can easily show that $\hat{h}$, $\hat{p}$ and $\hat{b}$ satisfy the Poincaré algebra (3). Furthermore, in the spinless sector of $\mathcal{F}_N$, for example $\{f_{\mathrm{sym}}(z_1, \ldots, z_N) \otimes v_+ \otimes \cdots \otimes v_+\}$ with f_{sym} being symmetric functions, $\hat{h}$, $\hat{p}$ as well as all the integrals of motion (19) of the model coincide with those of the relativistic Calogero–Sutherland model (1) and (2). This is due to the following formula [6] valid on this sector,

$$D_k(p^{\pm 1}, t^{\pm 1}) = (-t^{1/2})^{\pm k(N-1)} \sum_{i_1 < \cdots < i_k} y_{i_1}^{\pm 1} \ldots y_{i_k}^{\pm 1}$$

where we made identification $t = q^2$. From (5), this implies $H = \Delta^{1/2} \hat{h} \Delta^{-1/2}$ and $P = \Delta^{1/2} \hat{p} \Delta^{-1/2}$. We hence have obtained the integrable spin generalization of the relativistic Calogero–Sutherland model and shown that it possesses the quantum affine symmetry $U_q(\widehat{gl}_2)_0$.

We next consider the diagonalization of the spinless model and evaluate the dynamical correlation functions. The diagonalization of the integrals of motion (2) can be carried out by the Macdonald symmetric polynomials. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_N \geqslant 0$, $\lambda_j \in \mathbb{Z}$

be a partition and denote the Macdonald symmetric polynomial by $P_\lambda(z;\,p,t)$. Then one has [13]

$$D_k(p^{\pm 1}, t^{\pm 1})P_\lambda(z;\,p,t) = \left(\sum_{i_1<\cdots<i_k}\prod_{l=1}^{k} t^{N-i_l}p^{\lambda_l}\right)P_\lambda(z;\,p,t).$$

Therefore, from (5), we obtain the exact eigenvalues of H and P as

$$E_N(\lambda) = c^2\sum_{j=1}^{N}\cosh\frac{\theta_j}{c} \qquad P_N(\lambda) = c\sum_{j=1}^{N}\sinh\frac{\theta_j}{c} \tag{21}$$

$$\theta_j = \frac{2\pi}{L}\left\{\lambda_j + g\left(\frac{N+1}{2} - j\right)\right\} \tag{22}$$

where we set $\alpha = 2\pi/L$. The corresponding eigenfunctions are given by

$$\Psi_\lambda(z) = \Delta^{1/2}P_\lambda(z;\,p,t). \tag{23}$$

The model thus can be regarded as an ideal gas of N relativistic pseudo-particles with the pseudo-rapidities (22). One should note that formula (22) obeys the following Bethe ansatz-like equations

$$L\theta_j = 2\pi I_j + \pi(g-1)\sum_{l=1}^{N}\mathrm{sgn}(\theta_j - \theta_l) \tag{24}$$

with $I_j = \lambda_j + (N+1)/2 - j$.

The ground state is given by the function $\Psi_\phi(z) = \Delta^{1/2}$ corresponding to the empty partition $\lambda = \phi$. The ground-state momentum and energy eigenvalues are evaluated as $P_N^{(0)} = 0$ and

$$E_N^{(0)} = c^2\cosh\frac{\pi g N}{cL}\bigg/\sinh\frac{\pi g}{cL}. \tag{25}$$

Hence the ground state can be described as a filled Fermi sea with pseudo-momenta $P_j^{(0)} = c\sinh(\theta_j/c)$ with $-\theta_\mathrm{F} \leqslant \theta_j \leqslant \theta_\mathrm{F}$ $(j = 1, 2, \ldots, N)$, where $\theta_\mathrm{F} = \pi g(N-1)/L$.

The dynamical density–density correlation functions as well as the one-particle Green function can be evaluated by making use of the Macdonald symmetric polynomials. Here we summarize the results. To each partition λ, we assign a Young diagram $\mathcal{D}(\lambda) = \{(i,j)|1 \leqslant i \leqslant l(\lambda), 1 \leqslant j \leqslant \lambda_i,\ i,j \in \mathbb{Z}_{>0}\}$. Let λ' be the conjugate partition of λ. For each cell $\gamma = (i,j)$ of $\mathcal{D}(\lambda)$, we define the quantities $a(\gamma) = \lambda_i - j$, $a'(\gamma) = j - 1$, $l(\gamma) = \lambda'_j - i$ and $l'(\gamma) = i - 1$. Then we have

$$\langle 0|\rho(\xi,t)\rho(0,0)|0\rangle = \frac{2}{L^2}\sum_\lambda \frac{(1-p^{|\lambda|})^2(\chi^\lambda(p,t))^2}{h_\lambda(p,t)h_{\lambda'}(t,p)}\mathcal{N}(\lambda)\cos(\mathcal{P}(\lambda)\xi)\,\mathrm{e}^{-\mathrm{i}\mathcal{E}(\lambda)t} \tag{26}$$

$$\langle 0|\Psi^\dagger(\xi,t)\Psi(0,0)|0\rangle = \frac{A_N}{A_{N+1}}\sum_\lambda \frac{t^{2|\lambda|}((t^{-1})_\lambda^{(p,t)})^2}{h_\lambda(p,t)h_{\lambda'}(t,p)}\mathcal{N}(\lambda)\,\mathrm{e}^{-\mathrm{i}(\mathcal{E}(\lambda)t - \mathcal{P}(\lambda)\xi)} \tag{27}$$

with ξ being a real coordinate conjugate to the momentum P, $|\lambda| = \sum\lambda_j$, $\mathcal{E}(\lambda) = E_N(\lambda) - E_N^{(0)}$, $\mathcal{P}(\lambda) = P_N(\lambda)$ and

$$A_N = \prod_{j=1}^{N}\frac{(pt^{j-1};\,p)_\infty(t;\,p)_\infty}{(t^j;\,p)_\infty(p;\,p)_\infty}$$

$$h_\lambda(p,t) = \prod_{\gamma\in\lambda}(1 - p^{a(\gamma)}t^{l(\gamma)+1}) \qquad h_{\lambda'}(t,p) = \prod_{\gamma\in\lambda}(1 - p^{a(\gamma)+1}t^{l(\gamma)})$$

$$\mathcal{N}(\lambda) = \prod_{\gamma\in\lambda} \frac{1 - p^{a'(\gamma)}t^{N-l'(\gamma)}}{1 - p^{a'(\gamma)+1}t^{N-l'(\gamma)-1}}$$

$$\chi^{\lambda}(p,t) = \prod_{\substack{\gamma\in\lambda \\ \gamma\neq=(1,1)}} (t^{l'(\gamma)} - p^{a'(\gamma)}) \qquad (a)_{\lambda}^{(p,t)} = \prod_{\gamma\in\lambda}(t^{l'(\gamma)} - p^{a'(\gamma)}a).$$

For the rational coupling $g = r/s$, one should remark that the factor $\chi^{\lambda}(p,t)$ (respectively $(t^{-1})_{\lambda}^{(p,t)}$) vanishes if the diagram $\mathcal{D}(\lambda)$ contains the lattice point $(s+1, r+1)$ (respectively $(s, r+1)$). According to the same argument put forward by Ha in the CSM [15], this indicates that only the states which contain minimal r quasi-hole excitations accompanied by s (respectively $(s-1)$) quasi-particles can contribute as the intermediate states in (26) (respectively (27)). One can thus conclude that the excitations of the model obey the fractional exclusion statistics following Haldane [16] as in the CSM [15].

Furthermore, the exact spectra (21) allow one to analyse the finite-size scaling of the model in the thermodynamic limit, $N, L \to \infty$ with $N/L = n$ fixed. First of all, from (25) we obtain the finite-size correction to the ground-state energy as

$$\lim E_N^0 = L\varepsilon_0 - \frac{\pi v}{6L} g + \mathrm{O}\left(\frac{1}{L^2}\right) \tag{28}$$

where $\varepsilon_0 = (c^3/\pi g)\cosh(\pi g n/c)$ and $v = c\sinh(\pi g n/c)$ are the ground-state energy density and the velocity of the elementary excitation, respectively. In comparison with general theory [17], one may suspect that the central charge is given by g. However, this is not the correct identification [18]. The central charge should be identified with one. This can be justified by calculating the low-temperature expansion of the free energy from (24). Instead, we justify it here by deriving the whole conformal dimensions associated with the elementary excitations. These can be obtained by evaluating the differences of the total energy and momentum from the ground-state eigenvalues under change of the particle number (by ΔN) and transfer of the ΔD-particles from the left to the right Fermi point [18]. We hence obtain the finite-size corrections

$$\Delta E = \mu\Delta N + \frac{2\pi v}{L}\left[\frac{g}{4}\Delta N^2 + \frac{1}{g}\left(\Delta D + \frac{\Phi}{2\pi}\right)^2\right]$$

$$\Delta P = 2p_{\mathrm{F}}\Delta D + \frac{2\pi\cosh(\pi g n/c)}{L}\Delta N\left(\Delta D + \frac{\Phi}{2\pi}\right)$$

where $\mu = c^2\cosh(\pi g n/c)$ and $p_{\mathrm{F}} = v/g$ are the chemical potential and the Fermi momentum, respectively. We modified here the argument by Kawakami and Yang by considering the flux excitations Φ associated with the change of the particle number ΔN [19, 15]. Adding the contribution from the quasi-particle and quasi-hole excitations, we finally obtain the right and left conformal dimensions $h^{\pm}$ as follows,

$$h^{\pm}(\Delta N; \Delta D; N^{\pm}) = \frac{1}{2}\left[\frac{\sqrt{g}\Delta N}{2} \pm \frac{1}{\sqrt{g}}\left(\Delta D + \frac{\Phi}{2\pi}\right)\right]^2 + N^{\pm} \tag{29}$$

where $N^{\pm} \in \boldsymbol{Z}_{\geqslant 0}$. Remarkably, the result does not depend on c. The flux carried by a particle is πg as in the CSM [19] so that $\Phi = \pi g\Delta N$. One can thus write (29) as

$$h^{+} = \frac{1}{2g}(\Delta D + g\Delta N)^2 + N^{+} \qquad h^{-} = \frac{1}{2g}\Delta D^2 + N^{-}.$$

This result indicates that the effect of the flux excitation is equivalent to imposing the new selection rule $\Delta D = (g/2)\Delta N$ (mod 1) on (29) without $\Phi/2\pi$. This selection rule in fact can be obtained from the periodicity of the wavefunction of the pseudo-particle $\exp(\mathrm{i}\theta_j x_j)$ under the change $x_j \to x_j + L$. Hence $h^{\pm}$ with $N^{\pm} = 0$ can be regarded as the conformal dimensions of the $U(1)$-primary fields in the $C = 1$ Gaussian theory. From the results (26) and (27), we have also succeeded in obtaining the thermodynamic limit of the dynamical correlation functions and their low-energy asymptotic forms [20]. The critical exponents thus obtained agree with Ha's results [15] as well as those obtained from $h^{\pm}$ with assignment $\Delta N = 0$ for the density correlation and $\Delta N = 1$ for the one-particle Green function. We thus conclude that the model possesses the Tomonaga–Luttinger liquid property [21].

In the case with the special coupling $g = 2$, Gaussian theory is known to become the level-1 $su(2)$ Wess–Zumino–Witten theory. This feature is consistent with the results in [6], where setting $t = p^2$ is inevitable to define a new level-0 action of $U_q(\widehat{sl_2})$.

In comparison with the CSM, our model possesses one extra parameter c. The ultra-relativistic limit $c \to 0$ is especially interesting. There one has a decoupling of the left- and right-movers. In addition, the limit $g \to 0$ with g/c fixed reduces the Macdonald polynomial to the Hall–Littlewood function [13]. This suggests that a certain mathematical structure remains in this limit [22].

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After finishing this work, we found a paper by V Pasquier [23] where the same subject as in the first part of this paper is discussed.

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